Linear Algebra II

24/03/2017, Friday, 9:00 – 11:00

1 (10 + 10 = 20 pts) **Inner product spaces**

Consider the inner product space $C[-1, 1]$ with inner product

$$(f, g) := \int_{-1}^{1} f(t)g(t)dt.$$

Let $\mathcal{S}$ be the subspace of all linear functions of the form $g(t) = at + b$ with $a, b \in \mathbb{R}$.

(a) Determine an orthonormal basis of $\mathcal{S}$.

(b) Compute the best least squares approximation of the function $f(t) = e^t$ by a function from the subspace $\mathcal{S}$.

2 (4 + 5 + 6 + 10 = 25 pts) **Diagonalization**

We say that two $n \times n$ matrices are *simultaneously diagonalizable* if there exists a nonsingular $n \times n$ matrix S such that both $S^{-1}AS$ and $S^{-1}BS$ are diagonal (not necessarily identical).

(a) Let I denote the $n \times n$ identity matrix and let A be any diagonalizable $n \times n$ matrix. Show that I and A are simultaneously diagonalizable.

(b) Show that if A and B are simultaneously diagonalizable then $AB = BA$.

(c) Let D be a diagonal $n \times n$ matrix with n distinct entries on the diagonal. Find all $n \times n$ matrices that commute with D.

(d) Show that if $AB = BA$ and A has n distinct eigenvalues, then A and B are simultaneously diagonalizable.

3 $(5+5+5+5=20$ pts) **Hermitian matrices**

Let A be a $m \times m$ matrix, B a $n \times n$ matrix and C a $m \times n$ matrix. We consider the *Sylvester equation*

$$AX - XB = C$$

in the unknown $m \times n$ matrix X. Let $x_1, x_2 \ldots, x_n$ be the columns of X and $c_1, c_2 \ldots, c_n$ be the columns of C.

(a) First assume that B is a diagonal matrix with diagonal elements $b_1, b_2, \ldots, b_n$. Show that X satisfies the Sylvester equation if and only if $Ax_i - b_i x_i = c_i$ for $i = 1, 2 \ldots, n$.

(b) Show that the Sylvester equation has a solution X if, for $i = 1, 2, \ldots, n$, b_i is NOT an eigenvalue of A.

(c) Let U be a unitary matrix. Show that the Sylvester equation has a solution X if and only if the equation $AY - YUBU^T = CU^T$ has a solution Y.

(d) Now assume B is any Hermitian matrix, not necessarily diagonal. Show that the Sylvester equation has a solution X if A and B have no common eigenvalues.

4 $(20+5=25$ pts) **Singular value decomposition**

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) Find a singular value decomposition of A.

(b) Find the best rank 1 approximation for A.

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10 pts free